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LEAST SQUARES ESTIMATORS OF TWO INTERSECTING LINES

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Abstract

Least squares estimators are constructed for the slopes, β_1 and β_2, intercepts, α_1 and α_2, and point of intersection, X^*, of two straight lines. In constructing the estimators the residual sum of squares, S^2, is minimized in three stages. We assume that $X_1 < X_2 < \cdots < X_n$ are independent variables with the corresponding dependent variables $Y_1, \cdots, Y_n$ related to the X_i's by

$$EY_i = \alpha_1 + \beta_1 X_i \qquad \text{if } X_i \leq X^*$$

$$= \alpha_2 + \beta_2 X_i \qquad \text{if } X_i \geq X^*$$

The least squares estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{X}^*$ are found to satisfy

$$S^2(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{X}^*) = \min_{i=1,\cdots,n-1} \inf_{X_i < z < X_{i+1}} \inf_{\alpha_1,\cdots,\beta_2} \Big[\sum_{j=1}^{J(z)} (Y_i - \alpha_1 - \beta_1 X_i)^2 + \sum_{j=J(z)+1}^{n} (Y_i - \alpha_2 - \beta_2 X_i)^2 \Big]$$

where $J(z)$ = largest integer J such that $X_J < z$.

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In most standard applications of the method of least squares, the estimators which minimize the residual sum of squares are easily computed as solutions of a set of simultaneous linear equations. This is not the case when constructing estimators for the slopes, intercepts and point of intersection of two intersecting lines. The purpose of this note is to exhibit the somewhat more complicated minimization procedure which must be used in this case. The results below are evidently not new, though the author has been unable to find a discussion of this particular problem in the literature.

R. E. Quandt (1958 and 1960) discusses a similar estimation problem in which the two regression lines are not required to intersect as well as tests (assuming that the deviations from regression are normally distributed) of the hypothesis that the two regression lines are, in fact, the same. E. S. Page (1955 and 1957) discusses a non-parametric test of this hypothesis.

The problem under consideration here is as follows: Suppose $X_1 < X_2 < \cdots < X_n$ are a set of independent variables and $Y_1, \cdots, Y_n$ corresponding chance variables related to the X_i's by

$$\begin{aligned} EY_i &= \alpha_1 + \beta_1 X_i \qquad && \text{if } X_i \leq X^* \\ &= \alpha_2 + \beta_2 X_i && \text{if } X_i \geq X^* \end{aligned}$$

where $\beta_1 \neq \beta_2$ and $X^* = (\alpha_1 - \alpha_2)/(\beta_2 - \beta_1)$. We wish to compute least squares estimators of α_1, α_2, β_1, β_2 and X^*, i.e., we wish to find those numbers $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{X}$ such that

$$S^2 = \sum_{i=1}^{J(\hat{X}^*)} (Y_i - \hat{\alpha}_1 - \hat{\beta}_1 X_i)^2 + \sum_{i=J(\hat{X}^*)+1}^{n} (Y_i - \hat{\alpha}_2 - \hat{\beta}_2 X_i)^2$$

where $J(X^*)$ = the largest integer J such that $X_J \leq \hat{X}^*$ is a minimum. Note that we cannot simply defferentiate with respect to the five parameters since S^2 is not a differentiable function of X^*.

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S^2 is minimized in three stages as follows. Suppose first that X^* is known. To compute least squares estimators of the remaining parameters, say $\hat{\alpha}_1'$, $\hat{\alpha}_2'$, $\hat{\beta}_1'$, and $\hat{\beta}_2'$, S^2 is minimized subject to the restriction that $(\hat{\alpha}_1'-\hat{\alpha}_2')/(\hat{\beta}_2'-\hat{\beta}_1')=\hat{X}^*$. To do this we differentiate $S^2+\lambda[\alpha_1-\alpha_2-(\beta_2-\beta_1)X^*]$ with respect to α_1, α_2, β_1, β_2 and λ and equate to zero. This results in the equations

(1) $$0 = -2\sum_1^J (Y_i-\hat{\alpha}_1'-\hat{\beta}_1'X_i)+\lambda$$

(2) $$0 = -2\sum_{J+1}^n (Y_i-\hat{\alpha}_2'-\hat{\beta}_2'X_i)-\lambda$$

(3) $$0 = -2\sum_1^J (Y_i-\hat{\alpha}_1'-\hat{\beta}_1'X_i)X_i+\lambda X^*$$

(4) $$0 = -2\sum_{J+1}^n (Y_i-\hat{\alpha}_2'-\hat{\beta}_2'X_i)X_i-\lambda X^*$$

(5) $$0 = \hat{\alpha}_1'-\hat{\alpha}_2'+(\hat{\beta}_1'-\hat{\beta}_2')X^*$$

where $J=J(X^*)$. Equations (1),$\cdots$,(5) are readily solved, yielding the estimators

(6) $$\hat{\beta}_1' = D^{-1}\Big\{[\Sigma x_1y_1+K(\bar{y}_1-\bar{y}_2)(\bar{x}_1-X^*)]\,[\Sigma x_2^2+K(\bar{x}_2-X^*)^2]$$
$$+K[\Sigma x_2y_2+K(\bar{y}_2-\bar{y}_1)(\bar{x}_2-X^*)](\bar{x}_1-X^*)(\bar{x}_2-X^*)\Big\}$$

(7) $$\hat{\beta}_2' = D^{-1}\Big\{[\Sigma x_2y_2+K(\bar{y}_2-\bar{y}_1)(\bar{x}_2-X^*)]\,[\Sigma x_1^2+K(\bar{x}_1-X^*)]$$
$$+K[\Sigma x_1y_1+K(\bar{y}_1-\bar{y}_2)(\bar{x}_1-X^*)](\bar{x}_1-X\ )(\bar{x}_2-X^*)\Big\}$$

(8) $$\hat{\alpha}_2' = \frac{1}{\bar{x}_2-X^*}\left\{\frac{\Sigma x_2y_2}{n-J}+\bar{y}_2(\bar{x}_2-X^*)-\left[\frac{\Sigma x_2^2}{n-J}+\bar{x}_2(\bar{x}_2-X^*)\right]\hat{\beta}_2'\right\}$$

(9) $$\hat{\alpha}_1' = \hat{\alpha}_2'+(\hat{\beta}_2'-\hat{\beta}_1')X^*$$

where

$$K = \frac{J(n-J)}{n}$$

$$\bar{x}_1 = \frac{\sum_{i=1}^{J} X_i}{J}$$

$$\Sigma x_1^2 = \sum_{i=1}^{J} (X_i-\bar{x}_1)^2$$

$$\bar{y}_1 = \frac{\sum_{i=1}^{J} Y_i}{J}$$

$$\Sigma x_1 y_1 = \sum_{i=1}^{J} (X_i-\bar{x}_1)(Y_i-\bar{y}_1)$$

$$\bar{x}_2 = \frac{\sum_{i=J+1}^{n} X_i}{n-J} \quad ,$$

etc., and

$$D = [\Sigma x_1^2+K(\bar{x}_1-X^*)^2][\Sigma x_2^2+K(\bar{x}_2-X^*)^2]-[K(\bar{x}_1-X^*)(\bar{x}_2-X^*)]^2$$

Using the estimators of equations (6),...,(9) we find the minimum residual sum of squares given X^* to be

$$\begin{aligned} S^2(X^*) &= \Sigma y_1^2+\Sigma y_2^2 \\ &\qquad - \Big\{K[b_1(\bar{x}_2-X^*)(\Sigma x_2^2)^{-1}+b_2(\bar{x}_1-X^*)(\Sigma x_1^2)^{-1}]^2\Sigma x_1^2\Sigma x_2^2 \\ &\qquad +b_1\Sigma x_1^2+b_2\Sigma x_2^2+2K[b_1(\bar{x}_1-X^*)-b_2(\bar{x}_2-X^*)](\bar{y}_1-\bar{y}_2) \\ &\qquad -K(\bar{y}_1-\bar{y}_2)^2\Big\} \div \Big\{1+K[(\bar{x}_1-X^*)^2(\Sigma x_1^2)^{-1} \\ &\qquad +(\bar{x}_2-X\ \ )^2(\Sigma x_2^2)^{-1}]\Big\} \\ &= \Sigma y_1^2+\Sigma y_2^2-Q_1(X^*)/Q_2(X^*) \quad , \end{aligned}$$

say, where Q_1 and Q_2 are quadratic functions of X^*, and $b_i=\Sigma x_i y_i/\Sigma x_i^2$.

The second step in the minimization procedure is to suppose that J is known, that is, that it is known only that $X_J < X^* < X_{J+1}$, and to find the least squares estimator of X^*, say $\hat{X}^{*\prime}$, under this restriction. Thus we must find $\hat{X}^{*\prime}$ satisfying

$$S^2(\hat{X}^{*\prime}) = \inf_{X_J < z < J+1} S^2(z)$$

$$= \inf_{X_J < z < X_{J+1}} [\Sigma y_1^2 + \Sigma y_2^2 - Q_1(z)/Q_2(z)]$$

$$= \Sigma y_1^2 + y_2^2 - \sup_{X_J < z < X_{J+1}} [Q_1(z)/Q_2(z)]$$

In order to determine $\hat{X}^{*\prime}$, we proceed as follows. Write

$$Q_i(z) = r_i z^2 + s_i z + t_i \qquad (i=1,2)$$

and consider the ratio $Q_1(z)/Q_2(z)$ as a function of $z \in (-\infty, \infty)$. Note that

$$\frac{d}{dz}\frac{Q_1(z)}{Q_2(z)} = \frac{(r_2z^2+s_2z+t_2)(2r_1z+s_1)-(r_1z^2+s_1z+t_1)(2r_2z+s_2)}{(r_2z^2+s_2z+t_2)^2} = 0 \tag{10}$$

is satisfied at $\pm\infty$ and at the two roots, say z_1 and z_2, of the equation

$$(r_1s_2-s_1r_2)z^2+2(r_1t_2-t_1r_2)z+(s_1t_2-t_1s_2)=0 \quad .$$

Now since $Q_2(z) > 0$ for all z, the roots $\pm\infty$ are asymptotes of the curve Q_1/Q_2. The remaining roots are

$$z_1 = \frac{\bar{y}_1 - b_1\bar{x}_1 - \bar{y}_2 + b_2\bar{x}_2}{b_2 - b_1}$$

and

$$z_2 = \left\{ [(\Sigma x_1^2)^{-1} + (\Sigma x_2^2)^{-1}] \left[(b_1\bar{x}_2\sqrt{\Sigma x_1^2/\Sigma x_2^2} + b_2\bar{x}_1\sqrt{\Sigma x_2^2/\Sigma x_1^2})^2 \right.\right.$$

$$\left. + 2(b_1\bar{x}_1 - b_2\bar{x}_2)(\bar{y}_1 - \bar{y}_2) - (\bar{y}_1 - \bar{y}_2)^2 \right]$$

$$-[\bar{x}_1^2(\Sigma x_1^2)^{-1}+\bar{x}_2^2(\Sigma x_2^2)^{-1}](b_1\sqrt{\Sigma x_1^2/\Sigma x_2^2}+b_2\sqrt{\Sigma x_2^2/\Sigma x_1^2})^2$$

$$\div\Big\{[(\Sigma x_1^2)^{-1}+(\Sigma x_2^2)^{-1}][b_1^2\bar{x}_2\Sigma x_1^2(\Sigma x_2^2)^{-1}+b_2^2\bar{x}_1\Sigma x_2^2(\Sigma x_1^2)^{-1}$$

$$+b_1b_2(\bar{x}_1+\bar{x}_2)+(b_1-b_2)(\bar{y}_1-\bar{y}_2)]$$

$$-[\bar{x}_1(\Sigma x_1^2)^{-1}+\bar{x}_2(\Sigma x_2^2)^{-1}][b_1\sqrt{\Sigma x_1^2/\Sigma x_2^2}+b_2\sqrt{\Sigma x_2^2/\Sigma x_1^2}]^2\Big\}-z_1 \quad ,$$

at one of which the ratio is a maximum and at the other of which it is a minimum. (This follows from the fact that Q_1 and Q_2 are both polynomials of even degree so that the asymptotes at $+\infty$ and $-\infty$ are identical. Hence Q_1/Q_2 cannot have local maxima at both z_1 and z_2, since otherwise there would exist a point z_3 with $z_1<z_3<z_2$ at which the curve would be a local minimum. But then the derivative of Q_1/Q_2 would be zero at z_3, contradicting the fact that z_1 and z_2 are the only finite roots of equation (10). Similarly Q_1/Q_2 cannot have local minima at both z_1 and z_2.)

To determine whether $S^2(z_1)$ or $S^2(z_2)$ is the minimum sum of squares, note that

$$\lim_{z\to\pm\infty}\frac{r_1z^2+s_1z+t_1}{r_2z^2+s_2z+t_2}=\frac{r_1}{r_2} \quad ,$$

so that the value of $S^2(z)$ at the asymptotes is

$$\lim_{z\to\pm\infty}S^2(z)=\Sigma y_1^2+\Sigma y_2^2-\frac{Kb_2^2\frac{\Sigma x_2^2}{\Sigma x_1^2}+Kb_1^2\frac{\Sigma x_1^2}{\Sigma x_2^2}+2Kb_1b_2}{\frac{K}{\Sigma x_1^2}+\frac{K}{\Sigma x_2^2}}$$

$$=\Sigma y_1^2+\Sigma y_2^2-\frac{(\Sigma x_1y_1+\Sigma x_2y_2)^2}{\Sigma x_1^2+\Sigma x_2^2} \quad .$$

But the reduction in sum of squares at z_1 is

$$\frac{(\Sigma x_1y_1)^2}{\Sigma x_1^2}+\frac{(\Sigma x_1y_2)^2}{\Sigma x_2^2}>\frac{(\Sigma x_1y_1+\Sigma x_2y_2)^2}{\Sigma x_1^2+\Sigma x_2^2} \quad .$$

Thus for fixed J, $S^2(z)$ is a minimum in $(-\infty,\infty)$ at $z=z_1$.

Hence by the nature of the curve $Q_1(z)/Q_2(z)$, if consideration is restricted to $z\epsilon(X_J,X_{J+1})$, then $S^2(z)$ takes on its minimum value at $z=(\bar{y}_1-b_1\bar{x}_1-\bar{y}_2+b_2\bar{x}_2)/(b_2-b_1)$ if this point is in the interval (X_J,X_{J+1}) and at either X_J or X_{J+1} if not, i.e.,

$$\inf_{X_J<z<X_{J+1}} S^2(z)=S^2\left(\frac{\bar{y}_1-b_1\bar{x}_1-\bar{y}_2+b_2\bar{x}_2}{b_2-b_1}\right) \text{ if } X_J<\frac{\bar{y}_1-b_1\bar{x}_1-\bar{y}_2+b_2\bar{x}_2}{b_2-b_1}< X_{J+1}$$

$$= \min\left[S^2(X_J),S^2(X_{J+1})\right] \qquad \text{otherwise.}$$

The final step in the minimization procedure is to let J vary and minimize over the intervals (X_J,X_{J+1}). This minimization must be over $J=1,\cdots,n-1$, the values 1 and (n-1) being included to take into consideration the possibility that $X^*< X_2$ or $X^*>X_{n-1}$ (in which case estimators are obtained for only one of the lines). Thus the least squares estimators $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{X}^*$ satisfy

$$\min_{1\leq J\leq n-1}\ \inf_{X_J<X^*<X_{J+1}}\ \inf_{\alpha_1,\cdots,\beta_2}\left[\sum_{i=1}^{J}(Y_i-\alpha_1-\beta_1X_i)^2+\sum_{i=J+1}^{n}(Y_i-\alpha_2-\beta_2X_i)^2\right]$$

$$=\sum_{i=1}^{J(\hat{X}^*)}(Y_i-\hat{\alpha}_1-\hat{\beta}_1X_i)^2+\sum_{J(X^*)+1}^{n}(Y_i-\hat{\alpha}_2-\hat{\beta}_2X_i)^2$$

The above results now enable us to write the solution in a relatively simple form. Denote

$$\bar{x}_{1,J}=\frac{1}{J}\sum_1^J X_i\ ,\ \bar{x}_{2,J}=\frac{1}{n-J}\sum_{J+1}^{n}X_i$$

$$\bar{y}_{1,J}=\frac{1}{J}\sum_1^J Y_i\ ,\ \bar{y}_{2,J}=\frac{1}{n-J}\sum_{J+1}^{n}Y_i$$

$$b_{1,J}=\frac{\sum_1^J(X_i-\bar{x}_{1,J})(Y_i-\bar{y}_{1,J})}{\sum_1^J(X_i-\bar{x}_{1,J})^2} \qquad J=2,\cdots,n-1$$

$$a_{1,J} = \bar{y}_{1,J} - b_{1,J}\bar{x}_{1,J} \qquad J=2,\cdots,n-1$$

$$a_{1,1}+b_{1,1}X_1 = Y_1$$

$$b_{2,J} = \frac{\sum_{J+1}^{n}(X_i-\bar{x}_{2,J})(Y_i-\bar{y}_{2,J})}{\sum_{J+1}^{n}(X_i-\bar{x}_{2,J})^2} \qquad J=1,\cdots,n-2$$

$$a_{2,J} = \bar{y}_{2,J}-b_{2,J}\bar{x}_{2,J} \qquad J=1,\cdots,n-2$$

$$a_{2,n-1}+b_{2,n-1}X_n = Y_n \quad ,$$

and

$$S_J^2 = S^2\left(\frac{a_{1,J}-a_{2,J}}{b_{2,J}-b_{1,J}}\right) \quad \text{if} \quad \frac{a_{1,J}-a_{2,J}}{b_{2,J}-b_{1,J}} \in [X_J, X_{J+1}]$$

$$= \min\,[S^2(X_J), S^2(X_{J+1})] \quad \text{otherwise}$$

for $J=2,\cdots,n-2$, with

$$S_1^2 = \sum_{i=2}^{n}(Y_i-\bar{y}_{2,1})^2 - b_{2,1}^2 \sum_{i=2}^{n}(X_i-\bar{x}_{2,1})^2$$

$$S_{n-1}^2 = \sum_{i=1}^{n-1}(Y_i-\bar{y}_{1,n-1})^2 - b_{1,n-1}^2 \sum_{i=1}^{n-1}(X_i-\bar{x}_{1,n-1})^2$$

The solution is then those numbers $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{X}^*$ such that

$$S^2(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\beta}_1,\hat{\beta}_2,\hat{X}^*) = \min_{i=1,\cdots,n-1} S_i^2$$

(Note that both lines are estimated only if

$$\min_{i=1,\cdots,n-1} S_i^2 = \min_{i=2,\cdots,n-2} S_i^2 \,.)$$

This procedure may be applied directly to compute least squares estimators for two intersecting polynomials (of the same or differing degree), two non-intersecting polynomials (Cf. Quandt, 1958), or, in fact, many intersecting or non-intersecting polynomials in any sort of combination. The calculations required in these cases become successively more complex.

It has not been possible to determine the properties of the above estimators when all parameters are unknown. If the deviations from the true regression are assumed to be normally distributed with common variance σ^2, however, the above least squares estimators are maximum likelihood and the residual mean square is the maximum likelihood estimator of σ^2. Though exact variances of the above estimators have not been obtained, conditional variances and covariances given X^* are easily computed since the estimators in the conditional case (given in equations (6),•••,(9)) are linear functions of $Y_1, \cdots, Y_n$. Estimates of these conditional variances and covariances can be constructed in the usual way using the "pooled" residual sum of squares minimized above. One is tempted to use these estimates also in the unconditional case.

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